Thin Layer Models for Electromagnetism

Marc Duruflé, Victor Péron, Clair Poignard

THIN LAYER MODELS FOR ELECTROMAGNETISM.

M. Duruflé[†,*] **V. Péron**[‡,*], **C. Poignard**[†,*]
[‡]LMAP CNRS UMR 5142, INRIA Bordeaux-Sud-Ouest, Team MAGIQUE-3D, Pau, France.
[†]INRIA Bordeaux-Sud-Ouest, Institut de Mathématiques de Bordeaux, CNRS UMR 5251 & Université de Bordeaux1, Talence, France.
[*]Email: victor.peron@inria.fr

Talk Abstract

We investigate a model diffraction problem of electromagnetic waves in a biological cell. The cell is modelled as a medium surrounded by a thin membrane, embedded in an ambient medium. We study a transmission problem with a thin layer in electromagnetism. We derive approximate transmission conditions in order to replace the thin layer by these conditions on the boundary of the interior domain. Our approach is essentially geometric and based on a suitable change of variables in the thin layer. Numerical simulations validate the theoretical results.

1 Introduction

1.1 The considered problem

Let Γ be a compact oriented surface of $\mathbb{R}^3$ without boundary. Consider the smooth connected bounded domain $\mathcal{O}_{\mathsf{c}}$ enclosed by Γ; $\mathcal{O}_{\mathsf{c}}$ is surrounded by a thin layer $\mathcal{O}_{\mathsf{m}}^{\varepsilon}$ with constant thickness ε. This domain with thin layer is embedded in an ambient smooth connected domain $\mathcal{O}_{\mathsf{e}}^{\varepsilon}$ with compact oriented boundary. We denote by $\mathcal{O}$ the ε-independent domain defined by

$$\mathcal{O} = \mathcal{O}_{\mathsf{e}}^{\varepsilon} \cup \overline{\mathcal{O}_{\mathsf{m}}^{\varepsilon}} \cup \mathcal{O}_{\mathsf{c}}.$$

Moreover, we denote by Γ_ε the boundary of $\overline{\mathcal{O}_{\mathsf{c}} \cup \mathcal{O}_{\mathsf{m}}^{\varepsilon}}$ (see Fig. 1). Let μ_{c}, μ_{m}, and μ_{e} be three positive constants and let q_{e}, q_{c}, and q_{m} be three complex numbers. Define the two piecewise functions μ and q on $\mathcal{O}$ by :

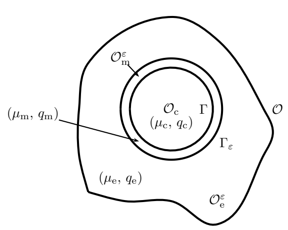

Figure 1: Geometry of the model

$$\forall x \in \mathcal{O}, \mu(x) = \begin{cases} \mu_{\mathsf{e}}, \text{ in } \mathcal{O}_{\mathsf{e}}^{\varepsilon}, \\ \mu_{\mathsf{m}}, \text{ in } \mathcal{O}_{\mathsf{m}}^{\varepsilon}, \\ \mu_{\mathsf{c}}, \text{ in } \mathcal{O}_{\mathsf{c}}, \end{cases} \quad q(x) = \begin{cases} q_{\mathsf{e}}, \text{ in } \mathcal{O}_{\mathsf{e}}^{\varepsilon}, \\ q_{\mathsf{m}}, \text{ in } \mathcal{O}_{\mathsf{m}}^{\varepsilon}, \\ q_{\mathsf{c}}, \text{ in } \mathcal{O}_{\mathsf{c}}. \end{cases}$$

The function μ is the dimensionless permeability of $\mathcal{O}$ while the function $q = \omega^2 \left(\epsilon - \mathrm{i}\frac{\sigma}{\omega}\right)$ denotes its dimensionless complex permittivity. Here, ω is the frequency, ϵ is the permittivity, and σ is the conductivity of the domain. We assume that the ambient medium is submitted to a current density $\mathbf{J}$. We denote by $\mathcal{O}_{\mathsf{e}}^{d_0}$ the set of points in $\mathcal{O}_{\mathsf{e}}^{\varepsilon}$ at the distance greater than d_0 of Γ. Throughout the paper the following hypothesis holds:

Hypothesis 1.1 (i) $q \in \mathrm{L}^\infty(\mathcal{O})$*, and for all* $x \in \mathcal{O}$*,*

$$\Im(q(x)) < 0 < \Re(q(x)). \tag{1}$$

(ii) $\mathrm{supp}(\mathbf{J}) \Subset \mathcal{O}_{\mathsf{e}}^{d_0}, \quad \mathbf{J} \in \mathrm{L}^2(\mathcal{O}), \quad \mathrm{div}\,\mathbf{J} = 0,$ *in* $\mathcal{O}$.

Maxwell equations describe the behavior of the electromagnetic field in $\mathcal{O}$. Denote by $\mathbf{E}$ and $\mathbf{H}$ the vector fields representing respectively the electric and the magnetic fields in $\mathcal{O}$ in time-harmonic regime. Denote by $\mathbf{n}$ the normal vector field of $\partial\mathcal{O}$ outwardly directed from $\mathcal{O}$. Maxwell equations in time-harmonic regime write

$$\mathrm{curl}\,\mathbf{E} = \mathrm{i}\mu\mathbf{H}, \quad \mathrm{curl}\,\mathbf{H} = -\mathrm{i}\left(q\mathbf{E} + \mathbf{J}\right), \text{ in } \mathcal{O}, \tag{2}$$

and are complemented by the boundary condition $\mathbf{n} \times \mathbf{E} = 0$ on $\partial\mathcal{O}$. The aim of this paper is to derive transmission conditions equivalent to $\mathcal{O}_{\mathsf{m}}^{\varepsilon}$ in order to avoid its meshing.

1.2 Variational formulation.

Denote by $\mathbf{X_N}(\varepsilon) = \{\mathbf{E} \in \mathbf{H}_0(\mathrm{curl}, \mathcal{O}) |\ \ q_\varepsilon\mathbf{E} \in \mathbf{H}(\mathrm{div}, \mathcal{O})\}$ the functional space. Define the sesquilinear form a_ε in $\mathbf{X_N}(\varepsilon)$ adapted to a regularized variational formulation of the problem (2) by

$$a_\varepsilon(\mathbf{E}, \mathbf{E}_*) = \int_{\mathcal{O}} (\mu_\varepsilon^{-1}\,\mathrm{curl}\,\mathbf{E}\cdot\mathrm{curl}\,\overline{\mathbf{E}_*} + \mathrm{div}\, q_\varepsilon\mathbf{E}\,\mathrm{div}\,\overline{q_\varepsilon\mathbf{E}_*} - q_\varepsilon\mathbf{E}\cdot\overline{\mathbf{E}_*})\mathrm{d}\mathbf{x}$$

For all $\varepsilon > 0$, the variational problem writes:

$(\mathbf{FV})_\varepsilon$: Find $\mathbf{E}_\varepsilon \in \mathbf{X_N}(\varepsilon)$ such that for all $\mathbf{E}_* \in \mathbf{X_N}(\varepsilon)$,

$$a_\varepsilon(\mathbf{E}_\varepsilon, \mathbf{E}_*) = \int_{\mathcal{O}} \mathbf{F}_\varepsilon \cdot \overline{\mathbf{E}_*} - \mathrm{div}\,\mathbf{F}_\varepsilon\,\mathrm{div}\,\overline{q_\varepsilon\mathbf{E}_*}\mathrm{d}\mathbf{x}$$

Using Hypothesis 1.1 the following theorem holds: There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, there is at most one solution $\mathbf{E}_\varepsilon \in \mathbf{X_N}(\varepsilon)$ to the problem $(\mathbf{FV})_\varepsilon$, which satisfies

$$\|\operatorname{curl} \mathbf{E}_\varepsilon\|_{0,\mathcal{O}} + \|\operatorname{div} q_\varepsilon \mathbf{E}_\varepsilon\|_{0,\mathcal{O}} + \|\mathbf{E}_\varepsilon\|_{0,\mathcal{O}} \leqslant C \|\mathbf{F}_\varepsilon\|_{\mathbf{H}(\mathrm{div},\mathcal{O})}$$

with $C > 0$ independent of ε.

1.3 Approximate transmission conditions

Denote by $\mathcal{O}_\mathsf{e}$ the domain $\mathcal{O}_\mathsf{e} = \mathcal{O} \setminus \overline{\mathcal{O}_\mathsf{c}}$. Define $\tilde\mu$ and $\tilde q$ by

$$\forall x \in \mathcal{O}, \quad \tilde\mu(x) = \begin{cases} \mu_\mathsf{c}, \text{ in } \mathcal{O}_\mathsf{c}, \\ \mu_\mathsf{e}, \text{ in } \mathcal{O}_\mathsf{e}, \end{cases} \quad \tilde q(x) = \begin{cases} q_\mathsf{c}, \text{ in } \mathcal{O}_\mathsf{c}, \\ q_\mathsf{e}, \text{ in } \mathcal{O}_\mathsf{e}, \end{cases}$$

and $\tilde\kappa = \tilde\mu \tilde q$. Then, $\mathbf{E}_\varepsilon$ possesses the following asymptotic expansion, see [1], [3] for precise estimates

$$\mathbf{E}_\varepsilon \approx \mathbf{E}_0 + \varepsilon \mathbf{E}_1 + \cdots \quad \text{in} \quad \mathcal{O}_\mathsf{c} \cup \mathcal{O}_\mathsf{e}$$

$$\mathbf{E}_\varepsilon \approx \mathcal{E}^{\mathsf{m},0}(y_\alpha, \frac{h}{\varepsilon}) + \varepsilon \mathcal{E}^{\mathsf{m},1}(y_\alpha, \frac{h}{\varepsilon}) + \cdots \quad \text{in} \quad \mathcal{O}^\varepsilon_\mathsf{m}$$

Here (y_α, h) is a *normal coordinate system* in $\mathcal{O}^\varepsilon_\mathsf{m}$, [2], and $\mathbf{E}_0$ and $\mathbf{E}_1$ satisfy the following BVP

$$\begin{aligned}
&\operatorname{curl}\operatorname{curl} \mathbf{E}_0 - \tilde\kappa^2 \mathbf{E}_0 = \tilde\mu \mathbf{J} \quad \text{in} \quad \mathcal{O} \\
&\operatorname{curl}\left(\tilde\mu^{-1} \operatorname{curl} \mathbf{E}_1\right) - \tilde q\, \mathbf{E}_1 = 0 \quad \text{in} \quad \mathcal{O}_\mathsf{c} \cup \mathcal{O}_\mathsf{e} \\
&\left[\mathbf{n} \times \mathbf{E}_1 \times \mathbf{n}\right]|_\Gamma = \left(\frac{q_\mathsf{c}}{q_\mathsf{m}} - \frac{q_\mathsf{c}}{q_\mathsf{e}}\right) \nabla_\Gamma \left(\mathbf{E}_0|_{\Gamma-} \cdot \mathbf{n}\right) \\
&+ \mu_\mathsf{c}^{-1}(\mu_\mathsf{m} - \mu_\mathsf{e}) \left(\operatorname{curl} \mathbf{E}_0 \times \mathbf{n}\right)|_{\Gamma-}, \\
&\left[\tilde\mu^{-1} \operatorname{curl} \mathbf{E}_1 \times \mathbf{n}\right]|_\Gamma = (q_\mathsf{m} - q_\mathsf{e}) \left(\mathbf{n} \times \mathbf{E}_0 \times \mathbf{n}\right)|_{\Gamma+} \\
&+ \left(\mu_\mathsf{m}^{-1} - \mu_\mathsf{e}^{-1}\right) \operatorname{Curl}_\Gamma \operatorname{curl}_\Gamma \left(\mathbf{n} \times \mathbf{E}_0 \times \mathbf{n}\right)|_{\Gamma+}, \\
&\mathbf{E}_0 \times \mathbf{n} = \mathbf{E}_1 \times \mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{O}.
\end{aligned}$$

2 Illustrations

We have tested the previous model (§1.3) when Γ is a sphere of radius 0.04. The boundary of $\mathcal{O}$ is a sphere of radius 0.08. We impose a Silver-Muller condition on this outer boundary. The current source is a Gaussian source polarized along x-coordinate and centered around the point (0, 0, 0.06). The exact solution is computed numerically on a similar mesh, where a thin layer made of hexahedra is inserted between the two domains. Edge finite elements of fourth order (Nedelec's first family) are used with curved elements in order to approximate correctly the geometry. We have observed that the numerical error between fourth order and fifth order is below 0.1 %. We chose the biological electrical parameters :

$$\epsilon_m = 10,\ \epsilon_e = \epsilon_c = 80,\ \sigma_m = 10^{-5},\ \sigma_e = \sigma_c = 0.5$$

and the frequency is equal to 1.2 GHz. Hence, we chose ε negligible compared to the vacuum wave length $\lambda_0 = 2, 5.10^{-1}$. The numerical values of $\mathbf{E}_0$ and $\mathbf{E}_1$ are displayed in Fig. 2. We have displayed the convergence of

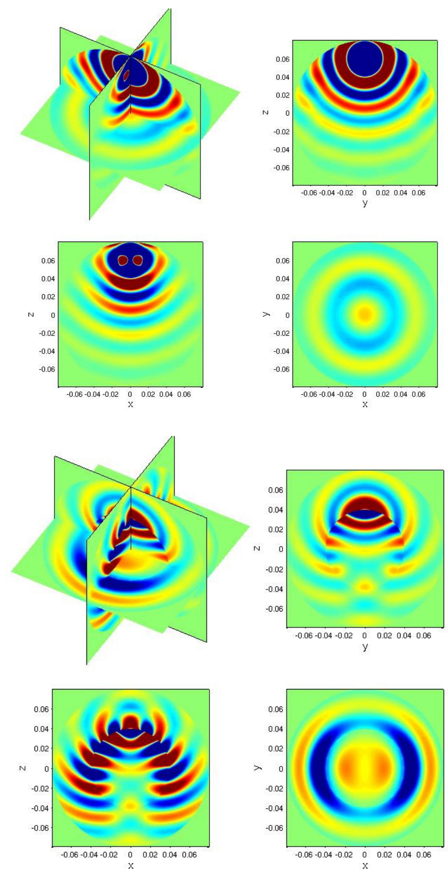

Figure 2: Real part of the electric field (x-component) for $\mathbf{E}_0$ (above) and $\mathbf{E}_1$ (below).

the model in Fig. 3. Observe that the numerical convergence rate coincide with the theory for small values of ε since it is in $O(\varepsilon^2)$, but not for the large values of ε. This is in accordance with the assumption "ε goes to zero", since at the crossingpoint of Fig. 3, ε equal 0.001 which is not small compared with the sphere radius of 0.04.

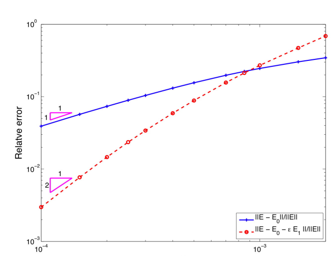

Figure 3: Relative error between the model and the exact solution.

In addition, the frequency range for which the thin layer model is valid has been studied. Actually, observe that in the biological parameters, the cell membrane conductivity is very low compared with the outer and inner conductivities, while the permittivity of the three domains are quite similar, compared with the membrane thickness. Moreover, for large frequency, the displacement currents are preponderant, meaning that the permittivities have to be mainly considered. Therefore, for large frequencies, the cell is a soft contrast material with a thin layer, and the theroretical results presented in this paper hold. However, if the frequency dramatically decreases, the conduction currents are dominating. In this case, the conductivities have to be used, and since the membrane conductivity is very low, the cell is then a high contrast medium with thin layer: two small parameters are then involved in the equation, and the asymptotic analysis presented here is no more valid. This phenomenon is illustrated by Fig. 4, where we have checked the accuracy of the model versus the frequency when ε is chosen constant, and equal to 0.0002: above 100MHz, the approximate transmission conditions precisely replace the membrane but below 10MHz, the conditions are no more valid and another analysis has to be performed, see section 3.

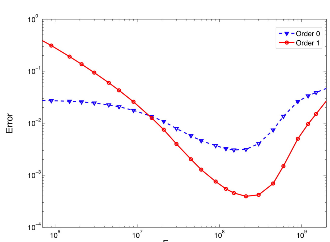

Figure 4: Relative error between the model and the exact solution versus frequency.

3 Resistive thin layer

In this section, we assume that $q = \varepsilon q_{\mathsf{m}}$ is of order ε in $\mathcal{O}^{\varepsilon}_{\mathsf{m}}$. Then, $\mathbf{E}_\varepsilon$ possesses the asymptotic expansion

$$\begin{aligned} \mathbf{E}_\varepsilon &\approx \mathbf{E}^{\mathsf{c}}_0 + \varepsilon \mathbf{E}^{\mathsf{c}}_1 + \cdots \quad \text{in} \quad \mathcal{O}_{\mathsf{c}} \\ \mathbf{E}_\varepsilon &\approx \widetilde{\mathbf{E}}^{\mathsf{e}}_0 + \varepsilon \widetilde{\mathbf{E}}^{\mathsf{e}}_1 + \cdots \quad \text{in} \quad \mathcal{O}^{\varepsilon}_{\mathsf{e}} \\ \mathbf{E}_\varepsilon &\approx \varepsilon^{-1} \mathcal{E}^{\mathsf{m},-1}(y_\alpha, \frac{h}{\varepsilon}) + \mathcal{E}^{\mathsf{m},0}(y_\alpha, \frac{h}{\varepsilon}) + \cdots \quad \mathcal{O}^{\varepsilon}_{\mathsf{m}} \end{aligned}$$

Hereafter, we present formal elements of proof of this asymptotic expansion.

3.1 Equations for the coefficients of the electric field

Integrating by parts in the variational formulation, we find the following Maxwell transmission problem for the electric field

$$\begin{array}{lll} \operatorname{curl}\operatorname{curl}\mathbf{E}^{\mathsf{e}}_\varepsilon - \kappa^2_{\mathsf{e}}\mathbf{E}^{\mathsf{e}}_\varepsilon = \mu_{\mathsf{e}}\mathbf{J} & \text{in} & \mathcal{O}^{\varepsilon}_{\mathsf{e}} \\ \operatorname{curl}\operatorname{curl}\mathbf{E}^{\mathsf{m}}_\varepsilon - \varepsilon\kappa^2_{\mathsf{m}}\mathbf{E}^{\mathsf{m}}_\varepsilon = 0 & \text{in} & \mathcal{O}^{\varepsilon}_{\mathsf{m}} \\ \operatorname{curl}\operatorname{curl}\mathbf{E}^{\mathsf{c}}_\varepsilon - \kappa^2_{\mathsf{c}}\mathbf{E}^{\mathsf{c}}_\varepsilon = 0 & \text{in} & \mathcal{O}_{\mathsf{c}} \\ \frac{1}{\mu_{\mathsf{e}}}\operatorname{curl}\mathbf{E}^{\mathsf{e}}_\varepsilon \times \mathbf{n} = \frac{1}{\mu_{\mathsf{m}}}\operatorname{curl}\mathbf{E}^{\mathsf{m}}_\varepsilon \times \mathbf{n} & \text{on} & \Gamma_\varepsilon \\ \frac{1}{\mu_{\mathsf{m}}}\operatorname{curl}\mathbf{E}^{\mathsf{m}}_\varepsilon \times \mathbf{n} = \frac{1}{\mu_{\mathsf{c}}}\operatorname{curl}\mathbf{E}^{\mathsf{c}}_\varepsilon \times \mathbf{n} & \text{on} & \Gamma \\ \mathbf{E}^{\mathsf{e}}_\varepsilon \times \mathbf{n} = \mathbf{E}^{\mathsf{m}}_\varepsilon \times \mathbf{n} & \text{on} & \Gamma_\varepsilon \\ \mathbf{E}^{\mathsf{m}}_\varepsilon \times \mathbf{n} = \mathbf{E}^{\mathsf{c}}_\varepsilon \times \mathbf{n} & \text{on} & \Gamma \\ \mathbf{E}^{\mathsf{e}}_\varepsilon \times \mathbf{n} = 0 & \text{on} & \partial\mathcal{O}. \end{array} \tag{3}$$

Here, $\kappa^2 = \mu q$ and the unit normal vector $\mathbf{n}$ on Γ is outwardly oriented to $\mathcal{O}_c$. Note that, since $\kappa \neq 0$, it is a consequence of the above equations that

$$\operatorname{div} q\mathbf{E}_\varepsilon = 0 \quad \text{in} \quad \mathcal{O}\,. \tag{4}$$

Therefore, the extra transmission condition holds

$$q_{\mathsf{e}}\mathbf{E}^{\mathsf{e}}_\varepsilon \cdot \mathbf{n} = \varepsilon q_{\mathsf{m}}\mathbf{E}^{\mathsf{m}}_\varepsilon \cdot \mathbf{n} \quad \text{on} \quad \Gamma_\varepsilon\,, \tag{5}$$

$$\varepsilon q_{\mathsf{m}}\mathbf{E}^{\mathsf{m}}_\varepsilon \cdot \mathbf{n} = q_{\mathsf{c}}\mathbf{E}^{\mathsf{c}}_\varepsilon \cdot \mathbf{n} \quad \text{on} \quad \Gamma\,. \tag{6}$$

We denote by $\mathbf{L}(y_\alpha, h; D_\alpha, \partial^h_3) = \operatorname{curl}\operatorname{curl} - \varepsilon\kappa^2_{\mathsf{m}}\mathbb{I}$, the 2nd order Maxwell operator set in $\mathcal{O}^{\varepsilon}_{\mathsf{m}}$. Here D_α is the covariant derivative on the interface Γ, and ∂^h_3 is the partial derivative with respect to the normal coordinate $y_3 = h$.

Let $a_{\alpha\beta}(h)$ be the metric tensor of the manifold Γ_h, which is the surface contained in $\mathcal{O}^{\varepsilon}_{\mathsf{m}}$ at a distance h of Γ. With this metric, a three-dimensional vector field $\mathbf{E}$ can be split into its normal component $\mathfrak{e}$ and its tangential component that can be viewed as a one-form field $\mathfrak{E}_\alpha$.

We denote by $\mathbf{B}(y_\alpha, h; D_\alpha, \partial^h_3)$ the tangent trace operator $\operatorname{curl}\cdot \times \mathbf{n}$ on $\Gamma \cup \Gamma_\varepsilon$ in a *normal coordinate system*. If $\mathbf{W} = (\mathfrak{E}_\alpha, \mathfrak{e})$, then

$$\Big(\mathbf{B}(y_\alpha, h; D_\alpha, \partial^h_3)\mathbf{W}\Big)_\alpha = \partial^h_3 \mathfrak{E}_\alpha - D_\alpha \mathfrak{e}\,,$$

see [2]. The operators $\mathbf{L}$ and $\mathbf{B}$ expand in power series of h with intrinsic coefficients with respect to Γ. We make the scaling

$$Y_3 = \varepsilon^{-1} h \tag{7}$$

to describe the behavior of the EM field in the thin layer with respect to ε. Then, the three-dimensional harmonic Maxwell operators in $\mathcal{O}^{\varepsilon}_{\mathsf{m}}$ writes $\mathbf{L}[\varepsilon]$. We define $\mathbf{B}[\varepsilon]$ the operator obtained from $\mathbf{B}$ on $\Gamma \cup \Gamma_\varepsilon$ after the scaling (7). These operators expand in power series of ε with coefficients intrinsic operators :

$$\mathbf{L}[\varepsilon] = \varepsilon^{-2} \sum_{n=0}^{\infty} \varepsilon^n \mathbf{L}^n \quad \text{and} \quad \mathbf{B}[\varepsilon] = \varepsilon^{-1}\mathbf{B}^0 + \mathbf{B}^1\,.$$

Assume that for $j \in \mathbb{N}$, the vector fields $\widetilde{\mathbf{E}}_j^{\mathsf{e}}$ are defined in $\mathcal{O}_{\mathsf{e}}$, and are as regular as necessary. Using formal Taylor expansion, we infer

$$\mathbf{E}_j^{\mathsf{e}} \times \mathbf{n}|_{h=\varepsilon^+} = \widetilde{\mathbf{E}}_j^{\mathsf{e}} \times \mathbf{n}|_{0^+} + \varepsilon\partial_h \widetilde{\mathbf{E}}_j^{\mathsf{e}} \times \mathbf{n}|_{0^+} + \cdots$$
$$\operatorname{curl} \mathbf{E}_j^{\mathsf{e}} \times \mathbf{n}|_{\varepsilon^+} = \operatorname{curl} \widetilde{\mathbf{E}}_j^{\mathsf{e}} \times \mathbf{n}|_{0^+} + \varepsilon\partial_h \operatorname{curl} \widetilde{\mathbf{E}}_j^{\mathsf{e}} \times \mathbf{n}|_{0^+} \cdots$$

It is convenient to define $\mathbf{E}_n$ for $n \in \mathbb{N}$ by $\mathbf{E}_n = \widetilde{\mathbf{E}}_n^{\mathsf{e}}$ in $\mathcal{O}_{\mathsf{e}}$, and $\mathbf{E}_n = \mathbf{E}_n^{\mathsf{c}}$ in $\mathcal{O}_{\mathsf{c}}$. Then according to the system (3), the profiles $\mathbf{W}_n = (\mathfrak{E}_n, \mathfrak{e}_n)$ and the terms $\mathbf{E}_n$ have to satisfy, for all $n \geqslant 0$:

$$\begin{aligned}
&\textit{(i)}\ \partial_3^2 \mathfrak{E}_{n-1,\alpha} = \sum_{j=0}^{n-1} L_\alpha^{n-j}(\mathbf{W}_{j-1}) \quad \text{in} \quad \Gamma \times I \\
&\textit{(ii)}\ \partial_3 \mathfrak{E}_{n-1,\alpha} = D_\alpha \mathfrak{e}_{n-2} + \tfrac{\mu_{\mathsf{m}}}{\mu_{\mathsf{e}}} \sum_{k=0}^{n-2} \partial_h^k \operatorname{curl} \widetilde{\mathbf{E}}_{n-k-2}^{\mathsf{e}} \times \mathbf{n}|_{0^+,\alpha} \\
&\textit{(iii)}\ \partial_3 \mathfrak{E}_{n-1,\alpha} = D_\alpha \mathfrak{e}_{n-2} + \tfrac{\mu_{\mathsf{m}}}{\mu_{\mathsf{c}}} \left(\operatorname{curl} \mathbf{E}_{n-2}^{\mathsf{c}} \times \mathbf{n}\right)_\alpha \\
&\textit{(iv)}\ \mathfrak{E}_{n-1} \times \mathbf{n} = \mathbf{E}_{n-1}^{\mathsf{c}} \times \mathbf{n} \quad \text{on} \quad \Gamma \times \{0\} \\
&\textit{(v)}\ \mathfrak{E}_{n-1} \times \mathbf{n} = \sum_{k=0}^{n-1} \partial_h^k \widetilde{\mathbf{E}}_{n-1-k}^{\mathsf{e}} \times \mathbf{n}|_{h=0^+} \quad \text{on} \quad \Gamma \times \{1\} \\
&\textit{(vi)}\ \operatorname{curl}\operatorname{curl} \widetilde{\mathbf{E}}_n^{\mathsf{e}} - \kappa_{\mathsf{e}}^2 \widetilde{\mathbf{E}}_n^{\mathsf{e}} = \delta_n^0 \mu_{\mathsf{e}} \mathbf{J} \quad \text{in} \quad \mathcal{O}_{\mathsf{e}} \\
&\textit{(vii)}\ \operatorname{curl}\operatorname{curl} \mathbf{E}_n^{\mathsf{c}} - \kappa_{\mathsf{c}}^2 \mathbf{E}_n^{\mathsf{c}} = 0 \quad \text{in} \quad \mathcal{O}_{\mathsf{c}} \\
&\textit{(viii)}\ \mathbf{E}_n^{\mathsf{e}} \times \mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{O}\ ,
\end{aligned} \tag{8}$$

with the convention $\mathbf{E}_n = 0$ when $n < 0$. Here, ∂_3 is the partial derivative with respect to Y_3 and $L_\alpha^1(\mathbf{W}) = -2b_\alpha^\beta \partial_3 \mathfrak{E}_\beta + \partial_3 D_\alpha \mathfrak{e} + b_\beta^\beta \partial_3 \mathfrak{E}_\alpha$, with $b_{\alpha\beta}$ the curvature tensor of the manifold Γ. The terms $\mathbf{W}_n$ and $\mathbf{E}_n$ are then determined by induction.

3.2 First terms of the asymptotics

According to the equations *(i) (ii) (iii)* in the system (8) for $n = 0$, $\mathfrak{E}_{-1}$ satisfies the ODE:

$$\left\{ \begin{array}{ll} \partial_3^2 \mathfrak{E}_{-1}(.,Y_3) & = 0 \quad \text{for} \quad Y_3 \in I\ , \\ \partial_3 \mathfrak{E}_{-1}(.,0) = \partial_3 \mathfrak{E}_{-1}(.,1) & = 0\ . \end{array} \right. \tag{9}$$

From the divergence free condition (4), we infer $\partial_3 \mathfrak{e}_{-1} = 0$. Therefore, the term $\mathbf{W}_{-1}$ depend only on y_α:

$$\mathbf{W}_{-1} = \mathbf{W}_{-1}(y_\alpha)\ . \tag{10}$$

From (5)-(6), we have: $q_{\mathsf{e}} \widetilde{\mathbf{E}}_0^{\mathsf{e}} \cdot \mathbf{n} = q_{\mathsf{m}} \mathfrak{e}_{-1}(.,1)$ and $q_{\mathsf{m}} \mathfrak{e}_{-1}(.,0) = q_{\mathsf{c}} \mathbf{E}_0^{\mathsf{c}} \cdot \mathbf{n}$. Equality (10) leads to

$$q_{\mathsf{e}} \widetilde{\mathbf{E}}_0^{\mathsf{e}} \cdot \mathbf{n} = q_{\mathsf{c}} \mathbf{E}_0^{\mathsf{c}} \cdot \mathbf{n} \quad \text{on} \quad \Gamma\ . \tag{11}$$

From *(iv)* and *(v)*, we have: $\mathfrak{E}_{-1}(.,0) = \mathfrak{E}_{-1}(.,1) = 0$. According to (10), we infer $\mathfrak{E}_{-1} = 0$ and $\mathbf{W}_{-1} = \mathfrak{e}_{-1}(y_\alpha)\mathbf{n}$.

The next term which is determined in the asymptotics is the profile $\mathbf{W}_0$. Since the terms $\partial_3 \mathbf{W}_{-1}$ and $\partial_3 \mathfrak{e}_{-1}$ vanish, from *(i)* when $n = 1$, we infer $\partial_3^2 \mathfrak{E}_0 = 0$ in I. Hence, $\partial_3 \mathfrak{E}_0$ is constant with respect to Y_3, and from *(ii)-(iii)*, we obtain

$$\partial_3 \mathfrak{E}_{0,\alpha} = \partial_\alpha \mathfrak{e}_{-1} \quad , Y_3 = 0, 1$$

We infer

$$\mathfrak{E}_{0,\alpha}(.,Y_3) = \mathfrak{E}_{0,\alpha}(.,0) + Y_3 \frac{q_{\mathsf{c}}}{q_{\mathsf{m}}} \partial_\alpha \mathbf{E}_0^{\mathsf{c}} \cdot \mathbf{n} \quad , Y_3 \in (0,1)$$

Hence, from *(iv)-(v)*, we obtain:

$$\widetilde{\mathbf{E}}_0^{\mathsf{e}} \times \mathbf{n} - \mathbf{E}_0^{\mathsf{c}} \times \mathbf{n} = \frac{q_{\mathsf{c}}}{q_{\mathsf{m}}} \partial_\alpha \mathbf{E}_0^{\mathsf{c}} \cdot \mathbf{n}|_{h=0^-}\ . \tag{12}$$

From the divergence free condition (4), we infer $\partial_3 \mathfrak{e}_0 = b_\alpha^\alpha \mathfrak{e}_{-1}$, $Y_3 \in I$, and $\mathfrak{e}_0(.,Y_3) = b_\alpha^\alpha \mathfrak{e}_{-1} Y_3 + \mathfrak{e}_0(.,0)$ when $Y_3 \in I$.

Since the terms $\mathfrak{E}_{-1} = 0$ and $\partial_3 \mathfrak{E}_{0,\alpha} = \partial_\alpha \mathfrak{e}_{-1}$, from *(i)* when $n = 2$ we infer $\partial_3^2 \mathfrak{E}_1 = -b_\alpha^\beta \partial_\beta \mathfrak{e}_{-1}$ in I. The right-hand side of the previous equality does not depend on Y_3, hence according to *(ii)* $-$ *(iii)* for $n = 2$ we infer the following transmission condition:

$$\frac{1}{\mu_{\mathsf{e}}} \operatorname{curl} \widetilde{\mathbf{E}}_0^{\mathsf{e}} \times \mathbf{n} = \frac{1}{\mu_{\mathsf{c}}} \operatorname{curl} \mathbf{E}_0^{\mathsf{c}} \times \mathbf{n} \quad \text{on} \quad \Gamma\ . \tag{13}$$

Finally, from *(vi)*, *(vii)*, and *(viii)*, $\mathbf{E}_0$ satisfies the BVP

$$\begin{aligned} &\operatorname{curl}\operatorname{curl} \mathbf{E} - \tilde{\mu}\tilde{q}\mathbf{E} = \tilde{\mu}\mathbf{J}, \quad \text{in } \mathcal{O}_e \cup \mathcal{O}_c \\ &\mathbf{E} \times \mathbf{n} = 0, \quad \text{on } \partial\mathcal{O}, \end{aligned}$$

with the transmission conditions (11)-(12)-(13) on Γ.

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